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COMPARISONS OF THE PERFORMANCES OF ESTIMATORS OF A BOUNDED NORMAL MEAN UNDER SQUARED-ERROR LOSS

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Comparisons of the performances of estimators of a bounded normal mean under squared-error loss

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Abstract

The problem of estimating a normal mean θ based on $X \sim \mathcal{N}(\theta, 1)$ when $|\theta| \leq m$ for a known $m > 0$ under squared-error loss is considered in this paper. Eight estimators are compared, namely, the maximum likelihood estimators (mle), three dominators of the mle obtained from Moors (1981, 1985), Charras (1979) and Charras and van Eeden (1991), two minimax estimators from Casella and Strawderman (1981), the Pitman estimator and Bickel's (1981) asymptotically-minimax estimator. Numerical as well analytical results are presented. In particular we show that the dominating estimators constructed by Charras and van Eeden are inadmissible and that, for $m \leq 1$, Moors' dominating estimator is Casella and Strawderman's minimax estimator with respect to a two-point least-favourable prior. We also show that, for $0 < m \leq m_1 \simeq 0.5204372$, the estimator $\delta_o(x) \equiv 0$ dominates the mle. Explicit expressions are given for the dominators of Moors, Charras and Charras and van Eeden. Asymptotic results are proved on the behaviour of these estimators when $m \to \infty$, results which can also be observed from our graphs.

Key Words: Bounded Normal Mean; Boundary Estimator; Minimaxity; Risk Function; Admissibility; Least-Favourable Prior; Approximate Minimaxity; Bayes Estimator; Squared-Error Loss

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1 Introduction

The problem considered in this paper is the estimation under squared-error loss of a normal mean θ based on $X \sim \mathcal{N}(\theta, 1)$ when $|\theta| \leq m$ for a known $m > 0$.

This estimating problem is considered by Casella and Strawderman (1981), by Bickel (1981), and by Gatsonis, MacGibbon and Strawderman (1987). Casella and Strawderman show that, when $0 < m \leq m_0 \simeq 1.056742$, there exists a unique minimax estimator of θ with respect to a symmetric two-point least-favourable prior on $\{-m, m\}$. They give an explicit expression for it and show it dominates the maximum likelihood estimator. They also give a class of minimax estimators for the case where $1.4 \leq m \leq 1.6$. These estimators are minimax with respect to a symmetric three-point prior on $\{-m, 0, m\}$. Bickel gives an estimator which is asymptotically minimax for $m \to \infty$ and Gatsonis, MacGibbon and Strawderman graphically compare these estimators and the Pitman estimator for several values of m.

The usual estimator of a normal mean is the maximum likelihood estimator (mle) which is, as is well-known, inadmissible for squared-error loss in our case. Dominators for the mle can be obtained from results of Charras (1979), Moors (1981, 1985) and Charras and van Eeden (1991). These authors consider estimation in restricted parameter spaces in a very general setting, give conditions for inadmissibility for squared-error loss and either give methods of constructing dominators (Moors and Charras and van Eeden) or prove the existence of dominators within a given class of estimators (Charras). Their conditions are satisfied for the bounded normal-mean problem and the purpose of this paper is to find explicit expressions for these dominators and compare them, analytically and graphically, with the mle, the Casella-Strawderman minimax estimators, Bickel's approximate minimax estimator and the Pitman estimator.

Casella and Strawderman show that, when $m \leq 1$, their minimax estimator dominates the mle. One of our analytic results shows that, also for $m \leq 1$, Moors' dominator of the mle of a bounded normal mean is the Casella-Strawderman minimax estimator, implying that this Moors dominator is admissible when $m \leq 1$. Another analytic result we have is that the dominators in the Charras-van Eeden class are all inadmissible. Finally, again analytically, we show that the estimator $\delta_o(x) \equiv 0$ (which we will call the "trivial estimator") dominates the mle when $0 < m \leq m_1 \simeq 0.5204372$.

Explicit expressions for the estimators are presented in Section 2. In that section we also give, for some of the estimators, asymptotic results concerning their behaviour as $m \to \infty$. Our numerical comparisons in the form of graphs are presented in Appendix B and discussed in Section 3. The proofs of the lemmas and theorems are given in Appendix A.

We know of only one other family of distributions for which Charras' (1979) and Moors' (1981, 1985) dominators have been obtained and compared. These results can be found in Perron (2003). He compares the mle with its Charras and its Moors dominators, as well as with the Pitman estimator and the Bayes estimator with respect to a prior proportional to $(p(1-p))^{-1}$, for the case where $X \sim$ Bin (n, p) when $p \in [a, 1-a]$ for a given $a \in (0, 1/2)$. He gives an algorithm for finding the Charras dominator.

2 Estimators for a bounded normal mean and their risk functions under squared-error loss

The problem of estimating a bounded normal mean based on $X \sim \mathcal{N}(\theta, 1)$ is a special case of the following problem: $(\mathcal{X}, \mathcal{A})$ is a measurable space and $\mathcal{P} = \{P_\theta, \theta \in D\}$ is a probability measure on $(\mathcal{X}, \mathcal{A})$, where D is a subset of the set of θ for which P_θ is a probabilty measure on $(\mathcal{X}, \mathcal{A})$. Further, $D \subset \Re^k$ is convex and closed. The problem is to find, for a given loss function, "good" estimators of θ based on a random vector $X \in \Re^n$ defined on $(\mathcal{X}, \mathcal{A})$, where $\delta(X)$ is an estimator if it satisfies $P_\theta(\delta(X) \in D) = 1$ for all $\theta \in D$. Many results concerning admissibility and minimaxity for such models have, for various loss functions, been obtained (see e.g. van Eeden (2006)).

The present section contains explicit expressions for each of the estimators of a bounded normal mean considered in this paper. Formulas for their risk functions for squared-error loss are also given.

2.1 The maximum likelihood estimator and its risk function

The mle of θ for our problem of estimating a bounded normal mean is given by

$$\delta^{\text{mle}}(X) = \begin{cases} -m & \text{if } X \leq -m \\ X & \text{if } -m < X < m \\ m & \text{if } X \geq m. \end{cases}$$

The risk function of this mle is given by

$$R(\theta, \delta^{\text{mle}}) = (-m-\theta)^2\Phi(-m-\theta) + \int_{-m-\theta}^{m-\theta} z^2\phi(z)dz + (m-\theta)^2\Phi(-m+\theta),$$

where $\phi(.)$ and $\Phi(.)$ are the pdf and cdf of standard normal distribution, respectively.

2.2 Casella and Strawderman's minimax estimators and their risk functions

Casella and Strawderman (1981) give conditions for a Bayes estimator to be minimax for estimating a bounded normal mean based on $X \sim \mathcal{N}(\theta, 1)$ with squared-error loss. They show that a two-point symmetric prior on $\{-m, m\}$ is least-favorable if $m \leq m_0 \simeq 1.056742$, implying that the corresponding Bayes estimator is minimax. This m_0 is the solution of $R(0, \delta^{\text{cs.2}}) - R(m, \delta^{\text{cs.2}}) = 0$, where $\delta^{\text{cs.2}}$ is the Bayes estimator which is given by

$$\delta^{\text{cs.2}}(X) = m \tanh(mX). \tag{2.1}$$

The authors show that this minimax estimator dominates the mle. They also give a class of minimax estimators for symmetric three-point priors as follows: for a three-point prior $\pi(0) = \alpha$ and $\pi(-m) = \pi(m) = (1-\alpha)/2$, the Bayes estimator under the squared error is given by

$$\delta^{\text{cs.3}}(X) = \frac{(1-\alpha) m \tanh(mX)}{1 - \alpha + \alpha \exp(m^2/2)/\cosh(mX)}\,. \tag{2.2}$$

Casella and Strawderman show that, if α and m satisfy

$$(m^2 - 1)(m^2 - 1 + \exp(m^2/2))^{-1} \le \alpha \le 2(2 + \exp(m^2/2))^{-1}, \tag{2.3}$$

and α is such that $R(0, \delta^{\text{cs.3}}) - R(m, \delta^{\text{cs.3}}) = 0$, then $\delta^{\text{cs.3}}$ is a minimax estimator of θ when $|\theta| \le m$. They find, numerically, that these two conditions are satisfied when $1.4 \le m \le 1.6$. The corresponding risk functions of these estimators under squared-error loss, given by

$$R(\theta, \delta^{\text{cs.2}}) = \int_{-\infty}^{\infty} (m \tanh(mx) - \theta)^2 \phi(x - \theta) dx$$

and

$$R(\theta, \delta^{\text{cs.3}}) = \int_{-\infty}^{\infty} \left(\frac{(1-\alpha) m \tanh(mx)}{1 - \alpha + \alpha \exp(m^2/2)/\cosh(mx)} - \theta \right)^2 \phi(x - \theta) dx$$

respectively, are easily obtained numerically.

2.3 Moors' dominating estimator of the mle and its risk function

Moors (1981, 1985) considers the problem described in the begining of this section and gives sufficient conditions for "boundary estimators" to be inadmissible for squared-error loss.

Here, a boundary estimator is an estimator which takes values on or near the boundary of D with positive probability for some $\theta \in D$. He assumes that the problem is invariant with respect to a finite group $G = (g_1, \ldots, g_p)$ of measure-preserving transformations from $\mathcal{X}$ to $\mathcal{X}$ and that the induced group $\tilde{G}$ is commutative and satisfies

$$\tilde{g}(ad_1 + bd_2) = a\tilde{g}(d_1) + b\tilde{g}(d_2) \text{ for all } d_1, d_2 \in D, \text{ all } \tilde{g} \in \tilde{G}.$$

He then constructs a random, closed, convex subset D_X of D with the property that an estimator δ for which $P_\theta(\delta(X) \notin D_X) > 0$ for some $\theta \in D$ is inadmissible. These sets D_X are defined as follows. Let p_θ be the density of P_θ with respect to a σ-finite measue ν defined on $(\mathcal{X}, \mathcal{A})$ and let

$$\alpha(X, \bar{g}_j(\theta)) = \frac{p_{\bar{g}_j(\theta)}(X)}{S(X;\theta)}, j = 1, \ldots, p,$$

when $S(X;\theta) = \sum_{j=1}^p p_{\bar{g}_j(\theta)}(X) > 0$. Further define

$$h_X(\theta) = \begin{cases} \sum_{j=1}^p \alpha(X, \bar{g}_j(\theta))\bar{g}_j(\theta) & \text{when } S(X;\theta) > 0 \\ \\ \theta & \text{when } S(X;\theta) = 0 \end{cases}$$

then D_X is the convex closure of the range of $h_X(\theta)$ and boundary estimators are dominated by their projection unto D_X.

For the problem of estimating a bounded normal mean under squared error loss, Moors' conditions are satisfied with $p = 2$, $g_1(x) = x$ and $g_2(x) = -x$ which gives $h_X(\theta) = \theta \tanh(\theta X)$, because $p_\theta(x) = 1/\sqrt{2\pi}\exp(-(x-\theta)^2/2)$. So the subset D_X of D is given by

$$D_X = (-m\tanh(m|X|), m\tanh(m|X|)),$$

which implies by Moors that any estimator δ for which

$$P_\theta(\delta(X) \notin (-m\tanh(m|X|), m\tanh(m|X|)) > 0 \text{ for some } \theta \in D$$

is inadmissible and is dominated by its projection unto D_X. Hence, Moors' dominating estimator of the mle is given by

$$\delta^{\text{mr}}(X) = \begin{cases} -m\tanh(m|X|) & \text{if } X \leq -m\tanh(m|X|) \\ X & \text{if } -m\tanh(m|X|) < X < m\tanh(m|X|) \\ m\tanh(m|X|) & \text{if } X \geq m\tanh(m|X|). \end{cases} \quad (2.4)$$

The following theorem shows that, for $m \leq 1$, Moors' dominating estimator of the mle is Casella and Strawderman's minimax estimator. We also obtain there a more explicit expression for this dominator for the case when $m > 1$. The proof of the theorem is given in Appendix A.

Theorem 2.1 *Moors' dominating estimator of the mle can also be written as*

(i) if $0 < m \leq 1$, $\delta^{\text{mr}}(X) = m\tanh(mX)$;

(ii) if $m > 1$, *then*

$$\delta^{\text{mr}}(X) = \begin{cases} m\tanh(mX) & \text{if } X \geq \xi(m) \text{ or } X \leq -\xi(m) \\ X & \text{if } -\xi(m) < x < \xi(m), \end{cases}$$

where $\xi(m)$, $r(m) < \xi(m) < m$, *is the unique root of* $u(x) = x - m\tanh(mx) = 0$, *for* $x > 0$, *and* $r(m) = \frac{1}{m}\ln[m + \sqrt{m^2 - 1}]$.

The risk function of δ^{mr} under squared error loss is given in the following theorem, which is an immediate consequence of Theorem 2.1.

Theorem 2.2 *The risk function of* δ^{mr} *under squared error is given by*

(i) if $m \leq 1$, *then*

$$R(\theta, \delta^{\text{mr}}) = \int_{-\infty}^{\infty} [m\tanh(m(z+\theta)) - \theta]^2 \phi(z)dz;$$

(ii) if $m > 1$, then

$$\begin{aligned} R(\theta, \delta^{\mathrm{mr}}) &= \int_{-\infty}^{-\xi(m)-\theta} [m \tanh(m(z+\theta)) - \theta]^2 \phi(z) dz \\ &+ \int_{-\xi(m)-\theta}^{\xi(m)-\theta} z^2 \phi(z) dz + \int_{\xi(m)-\theta}^{\infty} [m \tanh(m(z+\theta)) - \theta]^2 \phi(z) dz, \end{aligned}$$

where $\phi(z)$ is the standard normal density function.

2.4 Charras's and Charras and van Eeden's dominating estimators of the mle and their risk functions

Charras (1979) considers the problem as described in the beginning of this section. He gives, for squared-error loss, conditions for boundary estimators to be non-Bayes as well as conditions for them to be inadmissible, where a boundary estimator is, for him, an estimator δ for which $P_\theta(\delta(X) \in B) > 0$ for all $\theta \in D$ and B is the boundary of D. For the case where $k = 1$ and $\theta \in [a, b]$ for known $-\infty < a < b < \infty$, he gives conditions for the existence of classes of dominators of his boundary estimators. We use his approach to construct dominating estimators of the mle for the bounded-normal-mean problem. These results are presented in Section 2.4.2.

The inadmissibility results of Charras (1979) are published in Charras and van Eeden (1991), but his dominators are only mentioned there. Instead, Charras and van Eeden study a different class of dominators of Charras' boundary estimators and these Charras-van Eeden dominators are used, in Section 2.4.1 to obtain dominators for the mle for the bounded-normal-mean problem.

2.4.1 δ^{cve} and its risk function

Charras and van Eeden (1991) construct, for squared-error loss, a class of dominating estimators δ^{cve} for boundary estimators $\delta(X)$ of θ when $\theta \in [a, b]$ with $-\infty < a < b < \infty$, i.e., they suppose that δ is an estimator satisfying

$$\left.\begin{array}{lcl} P_\theta(\delta(X) = a) & > & 0 \\ P_\theta(\delta(X) = b) & > & 0. \end{array}\right\} \quad \text{for all } \theta \in [a, b].$$

They further suppose that, for each $\theta_o \in D$,

$$\lim_{\theta \to \theta_o} \int_{\mathcal{X}} |p_\theta(x) - p_{\theta_o}(x)| d\nu(x) = 0, \tag{2.5}$$

where p_θ is the density of P_θ with respect to the σ-finite measure ν.

The authors then show that there exist estimators of the form

$$\delta^{\text{cve}}(X) = \begin{cases} a + \varepsilon_1 & \text{if } \delta(X) \leq a \\ \delta(X) & \text{if } a < \delta(X) < b \\ b - \varepsilon_2 & \text{if } \delta(X) \geq b \end{cases} \tag{2.6}$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_1 + \varepsilon_2 \leq b - a$, which dominate δ.

The risk function of δ^{cve} is given by

$$\begin{aligned} R(\theta, \delta^{\text{cve}}) &= (a + \varepsilon_1 - \theta)^2 P_\theta(\delta(X) \leq a) + (b - \varepsilon_2 - \theta)^2 P_\theta(\delta(X) \geq b) \\ &+ \int_{\mathcal{X}} (\delta(x) - \theta)^2 I(a < \delta(x) < b) p_\theta(x) d\nu(x). \end{aligned}$$

This Charras-van Eeden result with $a = -m$ and $b = m$ clearly applies to our problem of dominating the mle of a bounded normal mean, where because of the symmetry of the problem, one can take $0 < \varepsilon_1 = \varepsilon_2 = \varepsilon \leq m$. This gives a class of dominators of the mle of a bounded normal mean for squared-error loss and using the results of Charras and van Eeden

(1991) one finds that any $\varepsilon \in (0, \varepsilon_o]$, where $\varepsilon_o = \min\left(m(8\Phi(-2m)/(1+2\phi(-2m)), m\right)$ gives a dominator of the mle. However, each one of these dominators δ^{cve} of the mle is inadmissible. This follows from Brown (1986)'s necessary condition for admissibility for squared-error loss in the estimation of the mean of an exponential-family distribution. He shows that an admissible estimator has to be non-decreasing and our estimator δ^{cve} is clearly not non-decreasing, while the $\mathcal{N}(\theta, 1)$ distribution is an exponential-family distribution. This inadmissibility result is summarized in the following theorem:

Theorem 2.3 *Let $X \sim \mathcal{N}(\theta, 1)$ with $|\theta| \leq m$ for a known positive m. Then the Charras-van Eeden dominators of the mle of θ are inadmissible for squared-error loss.*

We have not been able to find dominators for these inadmissible dominators of the mle and so will not consider them any further in this paper.

2.4.2 δ^{ch} and its risk function

In this section Charras' (1979) method of obtaining dominating estimators for his boundary estimators is presented and used to find dominators of the mle in the bounded-normal-mean problem.

Let δ be a Charras boundary estimator, then Charras considers the following class of estimators

$$\delta^{\text{t}}(X) = \begin{cases} a(t) & \text{if } \delta(X) \leq a(t) \\ \delta(X) & \text{if } a(t) < \delta(X) < b(t) \\ b(t) & \text{if } \delta(X) \geq b(t), \end{cases} \tag{2.7}$$

where $a(t)$ and $b(t)$, $t \in [0, 1]$ take values in $[a, b]$ with $a(0) = a$, $b(0) = b$, $a(1) = b(1)$, $a(t)$ non-decreasing and $b(t)$ non-increasing. He then gives sufficient conditions on the functions $a(t)$ and $b(t)$, on the distribution of X and of $\delta(X)$ and on the loss function for δ^{t} to dominate

δ. These conditions are given in Appendix A. Here, we give this domination result for the special case of the bounded-normal-mean when $a(t) = -m(1-t)$ and $b(t) = m(1-t)$, $t \in [0,1]$. Obviously, Charras' conditions are satisfied in the bounded-normal-mean case and his dominator of the mle can then be written as follows:

$$\delta^{\mathrm{ch}}(X) \;=\; \begin{cases} -m(1-t) & \text{if } X \leq -m(1-t) \\ X & \text{if } -m(1-t) < X < m(1-t) \\ m(1-t) & \text{if } X \geq m(1-t). \end{cases}$$

For simplicity of the proof, we let $\varepsilon = mt \in [0, m]$ and rewrite this dominator as follows:

$$\delta^{\mathrm{ch}}(X) \;=\; \begin{cases} -(m-\varepsilon) & \text{if } X \leq -(m-\varepsilon) \\ X & \text{if } -(m-\varepsilon) < \delta(X) < m-\varepsilon \\ (m-\varepsilon) & \text{if } X \geq m-\varepsilon. \end{cases} \qquad (2.8)$$

Its risk function is given by

$$\begin{aligned} R(\theta, \delta^{\mathrm{ch}}) \;&=\; (m-\varepsilon+\theta)^2 P_\theta(X \leq -(m-\varepsilon)) + (m-\varepsilon-\theta)^2 P_\theta(X \geq m-\varepsilon) \\ &+\; \int_{-(m-\varepsilon)}^{m-\varepsilon} (x-\theta)^2 \phi(x-\theta)dx, \end{aligned}$$

and the following theorem holds:

Theorem 2.4 *Let $X \sim N(\theta, 1)$ with $|\theta| \leq m$ for a known $m > 0$. Then $\{\delta^{\mathrm{ch}} : 0 < \varepsilon \leq \varepsilon'\}$ is a class of dominating estimators for δ^{mle}, where ε' is the unique root of $\psi(x) = 0$, where $\psi(x) = g(2m-x) + g(x) - 2x$ and $g(x) = 2x\Phi(-x)$.*

The proof of this theorem is given in Appendix A. It is Charras' proof applied to our special case and it goes as follows. First of all it is clear that, for all $\varepsilon \in (0, m]$, δ^{ch} dominates δ^{mle}

on $[-m+\varepsilon, m-\varepsilon]$. Further, by the symmetry of the problem, it is sufficient to look at the behaviour of the risk functions on $(m-\varepsilon, m]$. It is then shown that

$$\Delta(\theta, \varepsilon) = R(\theta, \delta^{\mathrm{mle}}) - R(\theta, \delta^{\mathrm{ch}}) > 0 \text{ for } \varepsilon \in (0, m] \text{ and } \theta = m - \varepsilon,$$

$$\frac{\partial}{\partial \varepsilon}\Delta(\theta, \varepsilon)|_{\varepsilon=0} > 0$$

and

$$\psi(\varepsilon) = \min_{\theta \in (m-\varepsilon, m)} \frac{\partial}{\partial \varepsilon}\Delta(\theta, \varepsilon)$$

is strictly decreasing in ε with $\psi(0) > 0$ and $\psi(m) < 0$. And then the unique solution to $\psi(x) = 0$, $x \in [0, m]$ gives an ε' with the required property. But it should be noted that this ε' is a lower-bound on the set of ε' for which δ^{ch} dominates δ^{mle}.

2.5 The trivial estimator

For the estimator $\delta_o(x) \equiv 0$ the following theorem holds. Its proof is in Appendix A.

Theorem 2.5 *Let m_1 be the unique positive solution to $u(2m) + 1/2 - m^2 = 0$, where $u(x) = x^2\Phi(-x) - \Phi(-x) - x\phi(x)$. Then the estimator $\delta_o(x) \equiv 0$ dominates the mle if and only if* $0 < m \leq m_1 \simeq 0.5204372$. *Its risk function is given by θ^2.*

2.6 The Pitman estimator and its risk function

In this section we consider the Pitman estimator of θ defined as the Bayes estimator with respect to a uniform prior on $[-m, m]$ and squared-error loss. This Bayes estimator is the posterior mean of θ given X. Since the marginal probability density function of X is given by

$$p(X) = \int_{-m}^{m} p_\theta(X)\pi(\theta)d\theta = \frac{1}{2m}\left[\Phi(m-X) - \Phi(-m-X)\right],$$

the posterior probability density function of θ given X is

$$\begin{aligned} p(\theta|X) &= \frac{p_\theta(X)\pi(\theta)}{p(X)} \\ &= \frac{1}{\Phi(m-X)-\Phi(-m-X)}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(\theta-X)^2}{2}\right\}1_{\{|\theta|\le m\}}. \end{aligned}$$

Hence the Pitman estimator for θ is given by

$$\begin{aligned} \delta^{\mathrm{P}}(X) &= E(\theta|X) \\ &= X + \frac{\int_{-m-X}^{m-X} z\phi(z)dz}{\Phi(m-X)-\Phi(-m-X)} \\ &= X - \frac{\phi(m-X)-\phi(m+X)}{\Phi(m-X)-\Phi(-(m+X))}. \end{aligned}$$

Its risk function under squared-error loss is given by

$$R(\theta,\delta^{\mathrm{P}}) = \int_{-\infty}^{\infty}\left[z - \frac{\phi(m-(z+\theta))-\phi(-m-(z+\theta))}{\Phi(m-(z+\theta))-\Phi(-m-(z+\theta))}\right]^2\phi(z)dz,$$

which can be computed numerically.

2.7 Bickel's estimator and its risk function

Bickel (1981) gives a class of asymptotically minimax estimators for estimating a bounded normal mean. He constructs this class in the following way:
Let, for $|x|<1$, $\bar{\psi}(x)=\pi\tan\left(\frac{\pi}{2}x\right)$ and let

$$\psi_m(x) = \begin{cases} \bar{\psi}(x) & \text{if } |x|\le 1-a_m^2 \\ (\bar{\psi}(1-a_m^2)+\bar{\psi}'(1-a_m^2)(x^2-(1-a_m^2)))\mathrm{sgn}x & \text{if } |x|>1-a_m^2. \end{cases}$$

He then shows that an asymptotically minimax estimator δ^{B} is given by

$$\delta^{\mathrm{B}}(X) = X - \frac{1}{n}\psi_m\left(\frac{X}{n}\right),$$

where $n = m(1-a_m)^{-1}$, $a_m < 1$ and $ma_m \to \infty$ as $m \to \infty$.

Bickel (1981) suggests taking $a_m = m^{-\frac{1}{8}}$ which gives the following expression for $\psi_m(x)$:

$$\begin{cases} \pi \tan\left(\frac{\pi}{2}x\right) & \text{if } |x| \leq 1 - m^{-\frac{1}{4}} \\ (\pi \tan(\frac{\pi}{2}(1-m^{-\frac{1}{4}})) + \frac{\pi^2}{2}\sec^2(\frac{\pi}{2}(1-m^{-\frac{1}{4}}))(x-(1-m^{-\frac{1}{4}})))\text{sgn}x & \text{if } |x| > 1 - m^{-\frac{1}{4}}. \end{cases}$$

So Bickel's asymptotically minimax estimator for θ is given by

$$\delta^{\mathrm{B}}(X) = X - \frac{1-m^{-\frac{1}{8}}}{m}\psi_m\left(\frac{(1-m^{-\frac{1}{8}})X}{m}\right). \tag{2.9}$$

The corresponding risk function of $\delta^{\mathrm{B}}(X)$ under squared-error loss is given by

$$R(\theta, \delta^B) = 1 - \frac{\pi^2}{m^2} + o(m^{-2}). \tag{2.10}$$

2.8 The behaviour of the estimators for $m \to \infty$

Concerning the asymptotic behaviour of the estimators as $m \to \infty$, the following theorem holds. Its proof is given in Appendix A.

Theorem 2.6 *Let δ_m be one of the estimators δ^{mle}, δ^{mr}, δ^{ch}, δ^{P}, δ^{B}. Then $\delta_m(X) - X$ converges to 0 almost surely as $m \to \infty$.*

This theorem has the following obvious corollary.

Corollary 2.1 *Let δ_m and δ'_m be two of the estimators δ^{mle}, δ^{mr}, δ^{ch}, δ^{P}, δ^{B}. Then $\delta_m(X) - \delta'_m(X)$ converges to 0 almost surely as $m \to \infty$.*

3 Numerical comparisons

In this section we graphically compare the risk functions of the estimators δ^{mle}, $\delta^{\text{cs.2}}$, $\delta^{\text{cs.3}}$, δ^{mr}, δ^{ch}, δ^{P} and δ^{B} under squared-error loss. Figures 1 - 8 (in Appendix B) represent the risk functions of different estimators for different fixed values of m. To check the performance outside of the restricted parameter space, the risk functions are all plotted in a little wider interval: $[-\frac{5}{4}m, \frac{5}{4}m]$.

Figures 1-3 demonstrate the risk functions when $m = 0.5, 0.8$ and 1 respectively. From there we observe that the minimax estimator with respect to a two-point least-favourable prior of Casella and Strawderman (1981) has the same risk function as Moors' dominating estimator of the mle, a result that follows from our Theorem 2.1, where we show that these two are the same estimator. We also observe that Moors' dominating estimator of the mle and the one of Charras dominate the mle in these cases, as they should by Moors (1981, 1985) and by our Theorem 2.4. For $m = .5$, $R(\theta, \delta^{\text{mle}}) > \theta^2$, showing that the trivial estimator dominates the mle – as it should, by Theorem 2.5. When $m = 0.8$, δ^{ch} works better than δ^{mr} in the middle part of the parameter space but worse when $m = 1$. The Pitman estimator dominates the other estimators in the middle part of the parameter space but is worse at the boundary.

Figure 4 represents these risk functions when $m = 1.5$. Here a minimax estimator with respect to a three-point least favourable prior is also compared with the other estimators. We took (see (2.2) and (2.3)) $\alpha = 0.341$. We observe that the risk functions of δ^{mle}, δ^{mr} and δ^{ch} are very close to each other, which verifies our conclusions on the convergence modes of these estimators as in Section 2.8. The Bayes estimator with respect to a two-point least favourable prior ($\delta^{\text{cs.2}}$) performs very badly in the middle part of the restricted parameter space but very good at the boundary. But, as shown by Casella and Strawderman (1981), it is not minimax for $m > 1$. Conversely, the Pitman estimator performs much better than the other estimators in the middle part of the space but worse at the boundary.

Figures 5-6 illustrate the risk functions when $m = 1.8$ and 3, where it can be observed that δ^{mle}, δ^{mr} and δ^{ch} have almost identical risk functions. Note that $\delta^{\text{cs.3}}$ is not included in these graphs. We observe the same performance of the Pitman estimator and $\delta^{\text{cs.2}}$ as in Figure 4. Finally, Figures 7-8 plot the risk functions when $m = 5$ and 10. Moreover, Bickel's asymptotically minimax estimator is now included. In Figure 7, Bickel's asymptotically minimax estimator performs better than the other estimators in the middle of the restricted parameter space. However, when $m = 10$, we observe that the risk functions of all these estimators tend to be close to the constant risk function of the unrestricted maximum likelihood estimator, which again verifies the results shown in Section 2.8.

For each value of m there are two graphs with the right-hand side one giving a clearer picture of the differences between the risk functions near the boundary of the parameter space - thus giving a better idea of how robust the properties of the estimators are with respect to misspecification of m.

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A Appendix: Proofs of the Results in Section 2

In this section proofs are given for the results in Section 2.

1) **Proofs for Moors' dominator** δ^{mr}.

The following lemma is needed for the proof of Theorem 2.1.

Lemma A.1 *Let* $u(x) = x - m\tanh(mx)$ *and* $v(x) = x + m\tanh(mx)$*. Then*

(a) for $0 < m \le 1$*,* $u(x)$ *and* $v(x)$ *are increasing in* x *and have the same sign as* x*;*

(b) for $m > 1$*, let* $r(m) = \frac{1}{m}\ln[m + \sqrt{m^2 - 1}]$*. Then:*

(i) $u(x)$ *increases in* x *for* $x > r(m)$ *and for for* $x < -r(m)$*, and decreases for* $-r(m) < x < r(m)$*;*

(ii) $0 < r(m) < m$*;*

(iii) $u(r(m)) < 0$*;*

(iv) there exists a unique $\xi(m)$*,* $r(m) < \xi(m) < m$*, such that* $u(-\xi(m)) = u(\xi(m)) = 0$*.*

Proof

(a) For $0 < m \le 1$, since $u'(x) = 1 - m^2 sech^2(mx)$, we have

$$u'(x) > 0 \quad \Leftrightarrow \quad \exp(2mx) - 2m\exp(mx) + 1 = (e^{mx} - m)^2 + 1 - m^2 > 0.$$

Consequently, when $0 < m \le 1$, $u'(x) > 0$ for $x \in (-\infty, \infty)$ and, when $m = 1$, $u'(x) > 0$ for $x \ne 0$ and $u'(0) = 0$. So, $u(x)$ increases in x and $u(x)$ has the same sign as x because $u(0) = 0$. Since $v'(x) = 1 + m^2 sech^2(mx) > 0$, for $x \in (-\infty, \infty)$, we have $v(x)$ increases in x, and $v(x)$ has the same sign as x because $v(0) = 0$.

(b) (i) Since $u'(x) = 1 - m^2 sech^2(mx)$, we have

$$
\begin{aligned}
u'(x) > 0 \quad &\Leftrightarrow \quad \cosh(mx) > \frac{1}{m} \\
&\Leftrightarrow \quad |\exp(mx) - m| > \sqrt{m^2 - 1} \\
&\Leftrightarrow \quad \begin{cases} x > r(m) > 0 & \text{if } \exp(mx) > m \\ x < -r(m) < 0 & \text{if } \exp(mx) < m. \end{cases}
\end{aligned}
$$

So $u(x)$ increases in x when $x > r(m)$ and when $x < -r(m)$. It decreases in x when $-r(m) < x < r(m)$.

(ii) Let $p(x) = x - \ln[x + \sqrt{x^2 - 1}]/x$ for $x > 1$. Then $p(m) = m - r(m)$ for $m > 1$. Further note that

$$
p(x) = \frac{1}{x} \ln \left(\frac{\exp(x^2)}{x + \sqrt{x^2 - 1}} \right) > \frac{1}{x} \ln \left(\frac{\exp(x^2)}{2x} \right) > 0. \tag{A.1}
$$

Since $x > 1$, $x + \sqrt{x^2 - 1} < 2x$. So the first inequality in (A.1) holds. Let $q(x) = \exp(x^2) - 2x$. Because $q(1) = e - 2 > 0$ and $q'(x) = 2(x\exp(x^2) - 1) > 0$ for $x > 1$, we have $q(x) > 0$, that is, the second inequality in (A.1) also holds for $x > 1$. Hence, $p(x) > 0$ for $x > 1$, and so $0 < r(m) < m$ for $m > 1$.

(iii) Since

$$
\begin{aligned}
u(r(m)) &= r(m) - m \tanh(mr(m)) \\
&= \frac{1}{m} \ln(m + \sqrt{m^2 - 1}) - \sqrt{m^2 - 1},
\end{aligned}
$$

we have

$$
\begin{aligned}
u(r(m)) < 0 \quad &\Leftrightarrow \quad m\sqrt{m^2 - 1} > \ln(m + \sqrt{m^2 - 1}) \\
&\Leftrightarrow \quad f(m) > 0,
\end{aligned}
$$

where $f(x) = x\sqrt{x^2-1} - \ln(x+\sqrt{x^2-1})$ for $x > 1$. Since $f(1) = 0$ and

$$\begin{aligned} f'(x) &= \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} - \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right) \\ &= \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}} \\ &= 2\sqrt{x^2-1} > 0, \end{aligned}$$

for $x > 1$, we have $f(x)$ increases in x and $f(x) > f(0) = 0$, for $x > 1$. That is, $u(r(m)) < 0$ for $m > 1$.

(iv) By (i), $u(x)$ increases for $x > r(m)$ and for $x < -r(m)$. It decreases for $-r(m) < x < r(m)$. Since $u(r(m)) < 0$ (by (iii)) and $u(m) > 0$, by the continuity of $u(x)$, there exists the unique $\xi(m)$, $r(m) < \xi(m) < m$, such that $u(\xi(m)) = 0$. ♡

Proof of Theorem 2.1

(i) When $0 < m \leq 1$, it follows from Lemma A.1 that

$$x \leq -m\tanh(m|x|) \Leftrightarrow x \leq 0$$

and

$$x \geq m\tanh(m|x|) \Leftrightarrow x \geq 0$$

and this shows that, when $m \leq 1$, we can then rewrite (2.4) as $\delta^{\mathrm{mr}}(x) = m\tanh(mx)$ for $x \in (-\infty, \infty)$.

(ii) When $m > 1$, let $u(x) = x - m\tanh(mx)$. By Lemma A.1, $u(x)$ increases in x when $x > r(m)$ or when $x < -r(m)$. It decreases in x when $-r(m) < x < r(m)$. Moreover, $\xi(m)$ is the unique root of $u(x) = 0$ in $[r(m), m]$. Hence, $u(0) =$

$u(-m) = u(m) = 0$, $u(x) > 0$ when $-\xi(m) < x < 0$ or $x > \xi(m)$ and $u(x) < 0$ when $x < -\xi(m)$ and when $0 < x < \xi(m)$.

So, when $m > 1$,

$$x \leq -m \tanh(m|x|) \Leftrightarrow x \leq -\xi(m)$$

and

$$x \geq m \tanh(m|x|) \Leftrightarrow x \geq \xi(m).$$

This proves the result for the case where $m > 1$. ♡

2) **Proofs for the Charras dominator** δ^{ch}.

Charras (1979) (see also Charras and van Eeden (1991)) gives conditions for estimators of the form (2.7) to dominate a boundary estimator δ, i.e. an estimator δ satisfying

$$\left.\begin{array}{rcl} P_\theta(\delta(X) = a) & > & 0 \\ P_\theta(\delta(X) = b) & > & 0 \end{array}\right\} \text{ for all } \theta \in [a, b]. \tag{A.2}$$

Charras' conditions on $a(t)$ and $b(t)$ for (2.7) to dominate δ are

(a) $a(t)$ and $b(t)$ are continuous;

(b) $a(t)$ and $b(t)$ have continuous and bounded right derivatives which are bounded in absolute value on $[0, 1]$;

(c) $a(0) = a$, $b(0) = b$ and $a(1) = b(1)$;

(d) for all $t \in [a, b]$, $a'_+(t) = \dfrac{da(t)}{dt+} > 0$ and $b'_+(t) < 0$

and his conditions on the distribution of X and $\delta(X)$ and on the loss function are

(1) condition (2.5) is satisfied;

(2) The loss function $L(\theta, d)$ has, for all θ in a neighbourhood N of $[a, b]$, a partial derivative $\partial L/\partial d$ with respect to d which is, on $N \times N$, continuous in d and in θ. Moreover,

$$\frac{\partial L(\theta, d)}{\partial d} \begin{cases} < 0 & \text{when } d < \theta \\ = 0 & \text{when } d = \theta \\ > 0 & \text{when } d > \theta; \end{cases}$$

(3) The estimator δ to be dominated satisfies (A.2);

(4) The estimator δ has, for each $\theta \in [a, b]$, a Lebesgue density on (a, b), i.e. there exsits a function $f(y, \theta)$ such that, for all (α, β) with $a \leq \alpha < \beta \leq b$,

$$P_\theta(\alpha < \delta(X) < \beta) = \int_\alpha^\beta f(y, \theta) dy.$$

Moreover, that density is bounded on $(a, b) \times [a, b]$.

Clearly, these Charras conditions are satisfied for our bounded-normal-mean prolem.

REMARK: Charras also has results for the case where δ has a discrete distibution.

Our proof below of Theorem 2.4 is a special case of Charras' proof for his general case and we need the following lemmas A.2, A.3 and A.4 for our proof. The proofs of the lemmas A.2 and A.3 are straightforward and omitted.

Lemma A.2 *Let* $u(x) = x^2\Phi(-x) - \Phi(-x) - x\phi(x)$. *Then:*

(i) The risk function of δ^{mle} *is given by*

$$R(\theta, \delta^{\text{mle}}) = 1 + u(m + \theta) + u(m - \theta). \tag{A.3}$$

(ii) The risk function of δ^{ch} *is given by*

$$R(\theta, \delta^{\text{ch}}) = 1 + u(m - \varepsilon + \theta) + u(m - \varepsilon - \theta). \tag{A.4}$$

Lemma A.3 *Let $g(x) = u'(x) = 2x\Phi(-x)$. Then $g'(x) = 2(\Phi(-x) - x\phi(x))$, $g''(x) = 2(x^2 - 2)\phi(x)$ and the following properties of these functions hold:*

(i) $g''(x) \geq 0$ if and only if $|x| \geq \sqrt{2}$ and $g''(x) \to 0$ as $x \to \pm\infty$;

(ii) $g'(x)$ increases in x if and only if $|x| > \sqrt{2}$; $g'(x)$ attains its maximum at $x = -\sqrt{2}$ and its minimum at $x = \sqrt{2}$ and $g'(0) = 1$. There is one unique root η_0 of $g'(x) = 0$, $\eta_0 \in (0, \sqrt{2})$, $g'(x) \to 2$ as $x \to -\infty$ and $g'(x) \to 0$ as $x \to +\infty$.

(iii) $g(x)$ has the same sign as x for $x \in (-\infty, +\infty)$; $g(x)$ increases in x if $x < \eta_0$ and decreases otherwise; $g(x)$ attains its maximum at $x = \eta_0$ and the unique root of $g(x) = 0$ is $x = 0$; $g(x) \to -\infty$ as $x \to -\infty$ and $g(x) \to 0$ as $x \to +\infty$.

Lemma A.4 *Let $h(x, \theta) = g(x + \theta) + g(x - \theta)$, where (see the lemmas A.2 and A.3) $g(x) = u'(x) = 2x\Phi(-x)$ and $u(x) = x^2\Phi(-x) - \Phi(-x) - x\phi(x)$. Then*

(i) For fixed $\varepsilon \in (0, m)$,

$$\min_{\theta \in [m-\varepsilon, m]} h(m - \varepsilon, \theta) = h(m - \varepsilon, m) = g(2m - \varepsilon) + g(\varepsilon) - 2\varepsilon.$$

(ii) Let $\psi(x) = g(2m - x) + g(x) - 2x$ for $x \in [0, m]$. Then $\psi'(x) < 0$, $\psi(0) > 0$ and $\psi(m) < 0$, so there exists a unique root $\varepsilon' \in (0, m)$ of $\psi(x) = 0$ with $\psi(x) > 0$ for $0 \leq x < \varepsilon'$ and $\psi(x) < 0$ for $\varepsilon' < x \leq m$.

Proof of Lemma A.4

(i) Consider

$$\frac{\partial}{\partial \theta} h(m - \varepsilon, \theta) \quad = \quad g'(m - \varepsilon + \theta) - g'(m - \varepsilon - \theta).$$

For $\theta \in (m-\varepsilon, m]$, we have $m-\varepsilon+\theta > 0$ and $m-\varepsilon-\theta < 0$. So (see Lemma A.3) $g'(m-\varepsilon+\theta) < g'(0) = 1$ and $g'(m-\varepsilon-\theta) > g'(0) = 1$. Hence $g'(m-\varepsilon+\theta) - g'(m-\varepsilon-\theta) < 0$ and so $\frac{\partial}{\partial\theta}h(m-\varepsilon,\theta) < 0$. In other words, $h(m-\varepsilon,\theta)$ decreases as θ increases in $(m-\varepsilon, m]$, which implies that

$$\min_{\theta\in[m-\varepsilon,m]} h(m-\varepsilon,\theta) = h(m-\varepsilon, m).$$

(ii) Note that $h(m-\varepsilon, m) = g(2m-\varepsilon)+g(\varepsilon)-2\varepsilon = \psi(\varepsilon)$. Since $\psi'(x) = -2-g'(2m-x) + g'(x)$, with (see Lemma A.3) $g'(2m-x) > g'(\sqrt{2}) > -1$ and $g'(x) < 1$, we have $\psi'(x) < 0$ for $x \in [0, m]$.

♡

Proof of Theorem 2.4

As noted in Section 2.4.2, it is sufficient to prove that δ^{ch} dominates δ^{mle} on $(m-\varepsilon, m]$ for $0 < \varepsilon \leq \varepsilon'$.

Let

$$\Delta(\theta,\varepsilon) = R(\theta,\delta^{\text{mle}}) - R(\theta,\delta^{\text{ch}}),$$

then, by Lemma A.2,

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\Delta(\theta,\varepsilon) &= -\frac{\partial}{\partial\varepsilon}[u(m-\varepsilon+\theta)+u(m-\varepsilon-\theta)] \\ &= u'(m-\varepsilon+\theta)+u'(m-\varepsilon-\theta) \\ &= g(m-\varepsilon+\theta)+g(m-\varepsilon-\theta) \\ &= h(m-\varepsilon,\theta),\end{aligned}$$

where $h(x,\theta)$ and $g(x)$ are defined in Lemma A.4.

Then by Lemma A.4 (i), we have

$$\min_{\theta \in (m-\varepsilon, m]} h(m-\varepsilon, \theta) = h(m-\varepsilon, m) = \psi(\varepsilon) > 0,$$

for $\varepsilon \in (0, \varepsilon')$, where ε' is given by (ii) in Lemma A.4, implying that, for $0 \leq \varepsilon \leq \varepsilon'$,

$$\frac{\partial}{\partial \varepsilon} \Delta(\theta, \varepsilon) \geq h(m-\varepsilon, m) = \psi(\varepsilon) \geq 0.$$

But, $\Delta(m-\varepsilon, \varepsilon) > 0$ for all $\varepsilon \in (0, m]$, which proves the theorem. ♡

3) **Proof of Theorem 2.5**

By Lemma A.2

$$\Delta_o(\theta, m) = R(\theta, \delta^{\text{mle}}) - R(\theta, \delta_o) = u(m+\theta) + u(m-\theta) + 1 - \theta^2.$$

So, it needs to be shown that $u(2m) + 1/2 - m^2 = 0$ has a unique positive root m_1 and that

$$u(m+\theta) + u(m-\theta) + 1 - \theta^2 \begin{cases} \geq 0 & \text{for all } \theta \in [0, m] \\ > 0 & \text{for some } \theta \in [0, m] \end{cases}$$

if and only if $0 < m \leq m_1$.

First note that (see Lemma A.3)

$$\Delta_o(0, m) = 2u(m) + 1 > 0 \text{ for } m > 0.$$

Further, with $g(x) = u'(x) = 2x\Phi(-x)$,

$$\frac{\partial}{\partial \theta} \Delta_o(\theta, m) = g(m+\theta) - g(m-\theta) - 2\theta$$

and

$$\frac{\partial^2}{\partial\theta^2}\Delta_o(\theta,m) = g'(m+\theta) + g'(m-\theta) - 2,$$

so that

$$\frac{\partial}{\partial\theta}\Delta_o(\theta,m)|_{\theta=0} = 0 \text{ for all } m > 0$$

and (see Lemma A.3)

$$\frac{\partial^2}{\partial\theta^2}\Delta_o(\theta,m)|_{\theta=0} < 0, \text{ for all } 0 \le \theta \le m, m > 0$$

implying that $\Delta_o(\theta,m)$ is, for each $m > 0$, decreasing in $\theta \in [0,m]$.

A necessary and sufficient condition for δ_o to dominate δ^{mle} for a given $m > 0$ is therefore that $\Delta_o(m,m) \ge 0$. But

$$\Delta_o(m,m) = u(2m) + u(0) + 1 - m^2 = u(2m) + 1/2 - m^2$$

and this function has the following properties:

1) $\Delta_o(0,0) = u(0) + 1/2 = 0$;
2) $\frac{d}{dm}\Delta_o(m,m) = 2(g(2m) - m) = 2m(4\Phi(-2m) - 1)$.

So

$$\frac{d}{dm}\Delta(m,m)\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\}0 \Longleftrightarrow m\left\{\begin{array}{c} < \\ = \\ > \end{array}\right\}\frac{1}{2}\Phi^{-1}\left(\frac{3}{4}\right).$$

Further $\Delta_o(\sqrt{2}/2,\sqrt{2}/2) = u(\sqrt{2}) < 0$ and thus there exists a unique $m_1 > 0$ with

$$\Delta_o(m_1,m_1) = 0 \text{ and } \Delta_o(m,m) > 0 \text{ for } 0 < m < m_1,$$

which, together with the fact that $\Delta_o(\theta,m)$ is decreasing in θ for $\theta \in [0,m]$, proves the result. Numerically, we found $m_1 \simeq 0.5204372$. ♡

4) **Proof of Theorem 2.6**

It is sufficient to prove that, for each fixed $x \in (0, m)$, $\delta_m(x) - x \to 0$ as $m \to \infty$. This implies that $\delta^{\text{mle}}(x) - x$ converges to 0 almost surely as $m \to \infty$, so it is now sufficient to prove, for each δ_m that, for fixed $x \in (0, m)$, $\delta_m(x) - x \to 0$ or $\delta_m(x) - \delta^{\text{mle}}(x) \to 0$ as $m \to \infty$.

a) Proof for $\delta^{\text{mr}}(x) - \delta^{\text{mle}}(x)$:

For $x \in (0, m)$, $\delta^{\text{mr}}(x)$ differs from $\delta^{\text{mle}}(x)$ if and only if $0 < x < \xi(m)$, so it suffices to show that $\xi(m) \to \infty$ as $m \to \infty$. But this can be seen as follows. Since $\xi(m)$ is the unique positive root of $u(x) = x - m\tanh(mx)$, we have $u(\xi(m)) = 0$, that is, $\xi(m) = m\tanh(m\xi(m))$. Hence,

$$\begin{aligned} m - \xi(m) &= m - m\tanh(m\xi(m)) = \frac{2m\exp\left(-2m\xi(m)\right)}{1 + \exp(-2m\xi(m))} \\ &\to 0, \quad \text{as } m \to \infty. \quad \heartsuit \end{aligned}$$

b) Proof for $\delta^{\text{ch}}(x) - \delta^{\text{mle}}(x)$:

We know that, for fixed $x \in (0, m)$, $\delta^{\text{mle}}(x)$ and $\delta^{\text{ch}}(x)$ differ from each other if and only if $x \in (m - \varepsilon, m]$ where $\varepsilon \in (0, \varepsilon')$ with ε' the unique root of $\psi(x) = g(2m-x)+g(x)-2x = 0$ and $\psi(x)$ can also be written as $\psi(x) = g(2m-x)+g(-x)$. (see Theorem 2.4).

By the definition of ε' we have that $g(2m - \varepsilon') + g(-\varepsilon') = 0$. Since $0 < \varepsilon' < m < 2m$, $2m - \varepsilon' \to \infty$ as $m \to \infty$, implying that

$$g(-\varepsilon') = -2(2m - \varepsilon')\Phi(-(2m - \varepsilon')) \to 0 \text{ as } m \to \infty.$$

Consequenty, as $m \to \infty$, we have $g(-\varepsilon') \to 0$, which is equivalent to $\varepsilon' \to 0$ by the definition of $g(x)$. Hence, for $x \in (m-\varepsilon, m]$,

$$\delta^{\text{mle}}(x) - \delta^{\text{ch}}(x) = m - (m-\varepsilon) = \varepsilon < \varepsilon' \to 0 \quad \text{as } m \to \infty. \quad \heartsuit$$

c) Proof for $\delta^{\text{P}}(x) - x$:

As $m \to \infty$, $\int_{-m-x}^{m-x} z\phi(z)dz \to \int_{-\infty}^{\infty} z\phi(z)dz = 0$, while $\Phi(m-x) - \Phi(-m-x) \to 1$, hence, $\delta^{\text{P}}(x) - x \to 0$ as $m \to \infty$. $\heartsuit$

d) Proof for $\delta^{\text{B}}(x) - x$:

The difference between x and $\delta^{\text{B}}(x)$ is given by

$$x - \delta^{\text{B}}(x) = \frac{1-m^{-\frac{1}{8}}}{m}\psi_m\left(\frac{1-m^{-\frac{1}{8}}}{m}x\right).$$

As $m \to \infty$, we have

$$\frac{1-m^{-\frac{1}{8}}}{m} \to 0.$$

Further

$$\psi_m\left(\frac{1-m^{-\frac{1}{8}}}{m}x\right) \to 0,$$

because (see the definition of $\psi_m(x)$ in Section 2.7) for fixed $x > 0$ and large enough m,

$$\frac{1-m^{-\frac{1}{8}}}{m}x \le 1 - \frac{1}{m^{1/4}}$$

so that

$$\psi_m\left(\frac{1-m^{-\frac{1}{8}}}{m}x\right) = \pi \tan\left(\frac{\pi}{2}\frac{1-m^{-\frac{1}{8}}}{m}x\right) \to 0 \text{ as } m \to \infty. \quad \heartsuit$$

B Appendix: Graphs for Section 3

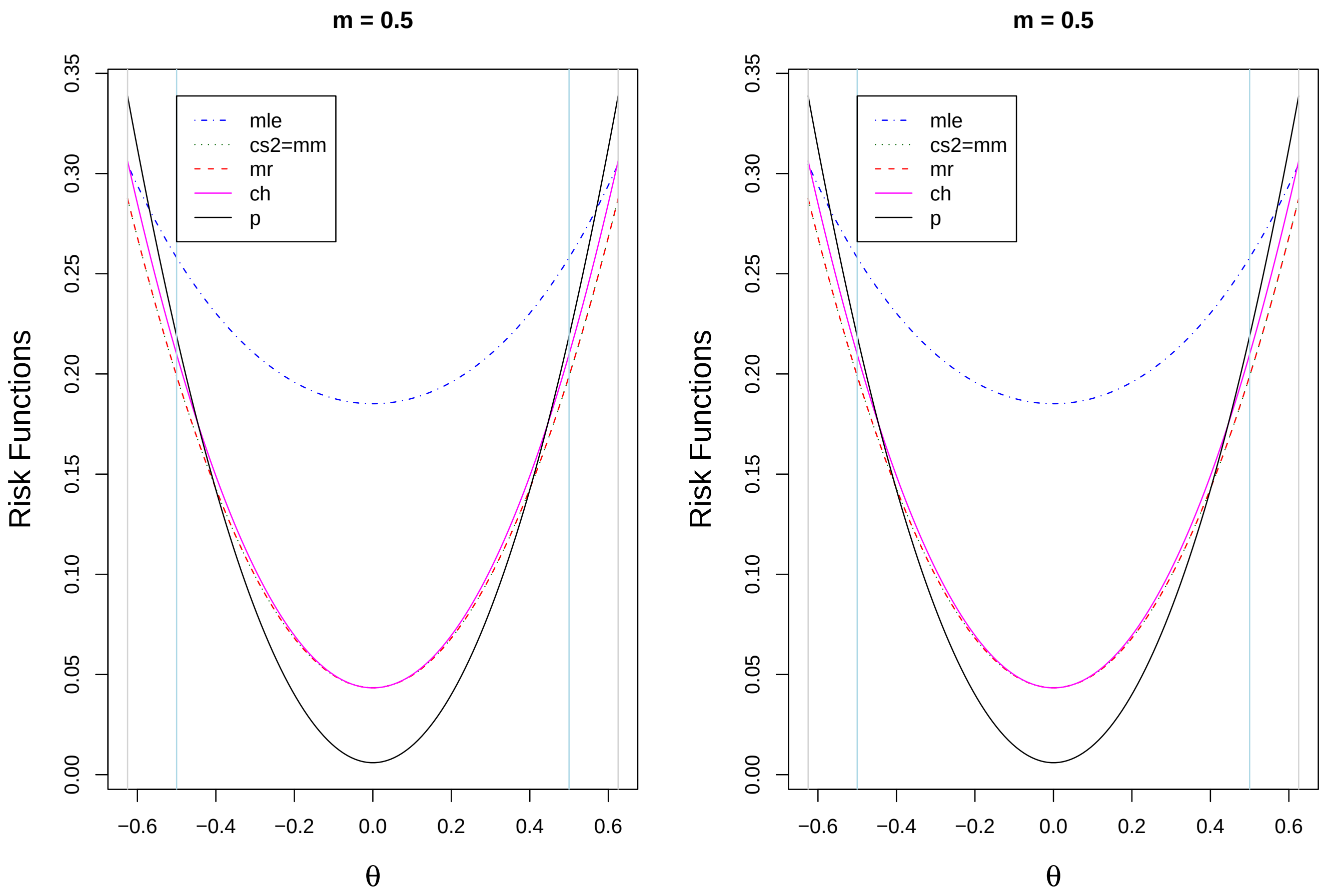

Figure 1: Comparisons of risk functions for $\delta^{\mathrm{mle}}, \delta^{\mathrm{cs.2}}, \delta^{\mathrm{mr}}, \delta^{\mathrm{ch}}$ and δ^{P}, at $m = 0.5$.

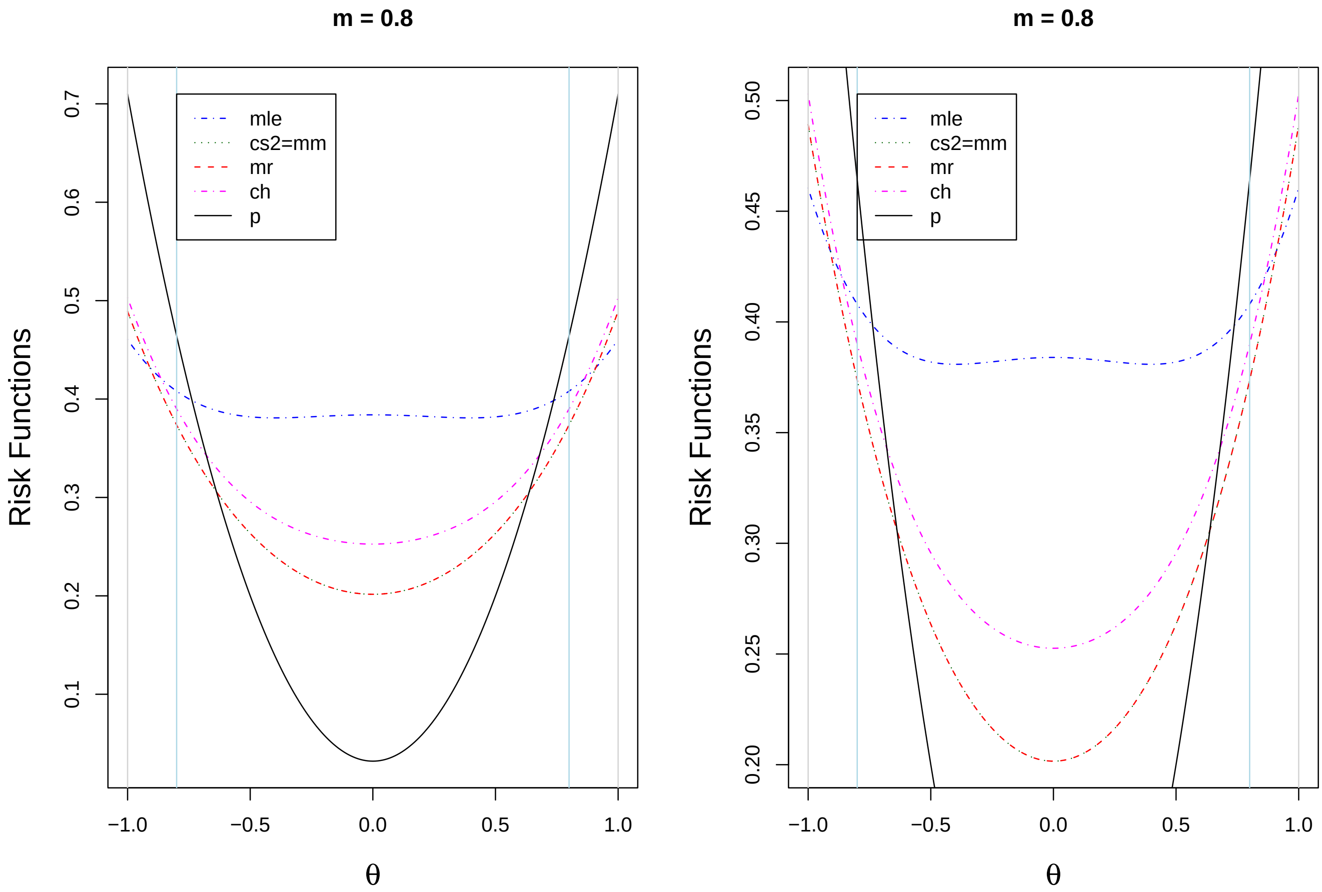

Figure 2: Comparisons of risk functions for $\delta^{\text{mle}}, \delta^{\text{cs.2}}, \delta^{\text{mr}}, \delta^{\text{ch}}$ and δ^{P}, at $m = 0.8$.

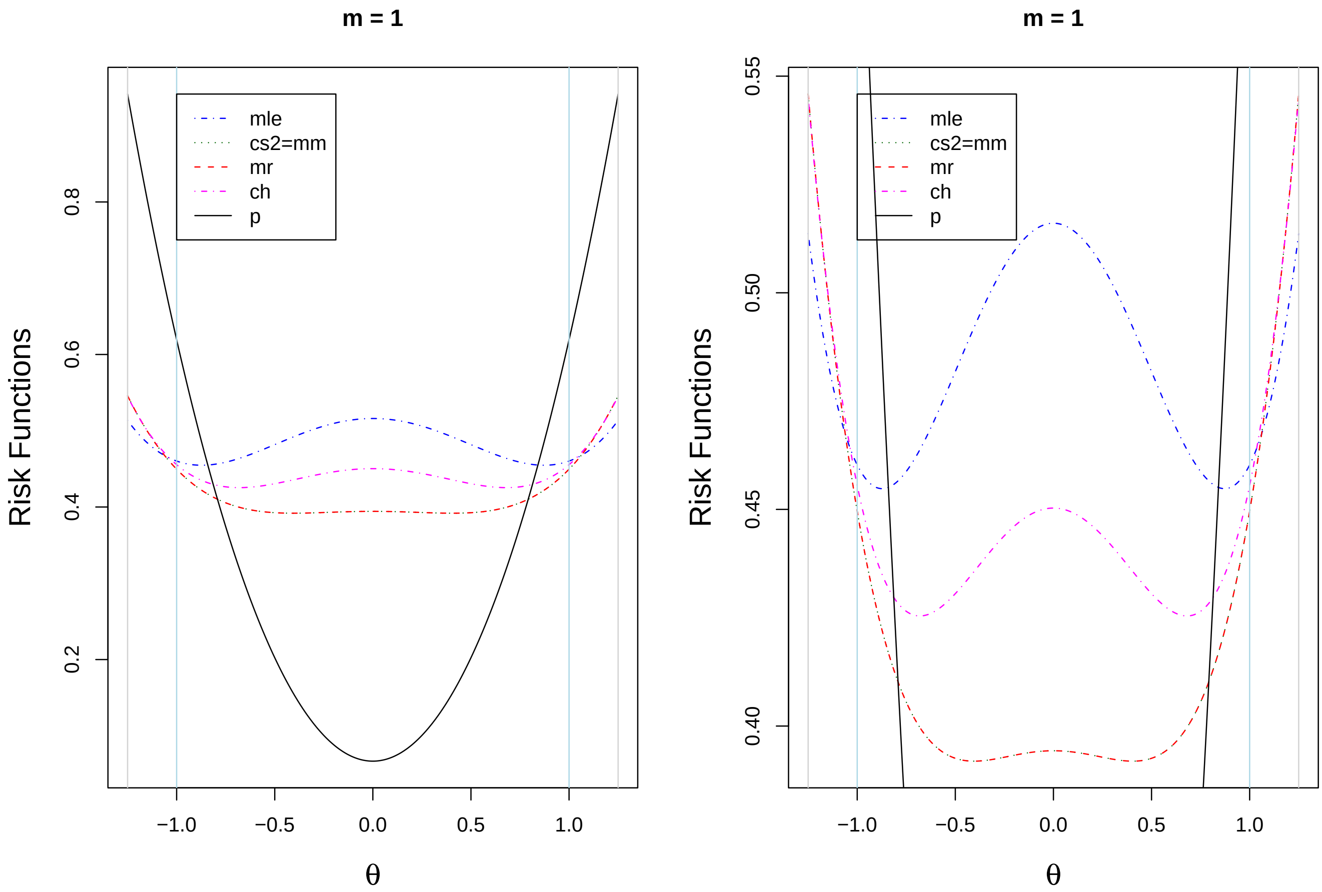

Figure 3: Comparisons of risk functions for $\delta^{\mathrm{mle}}, \delta^{\mathrm{cs.2}}, \delta^{\mathrm{mr}}, \delta^{\mathrm{ch}}$ and δ^{P}, at $m = 1$.

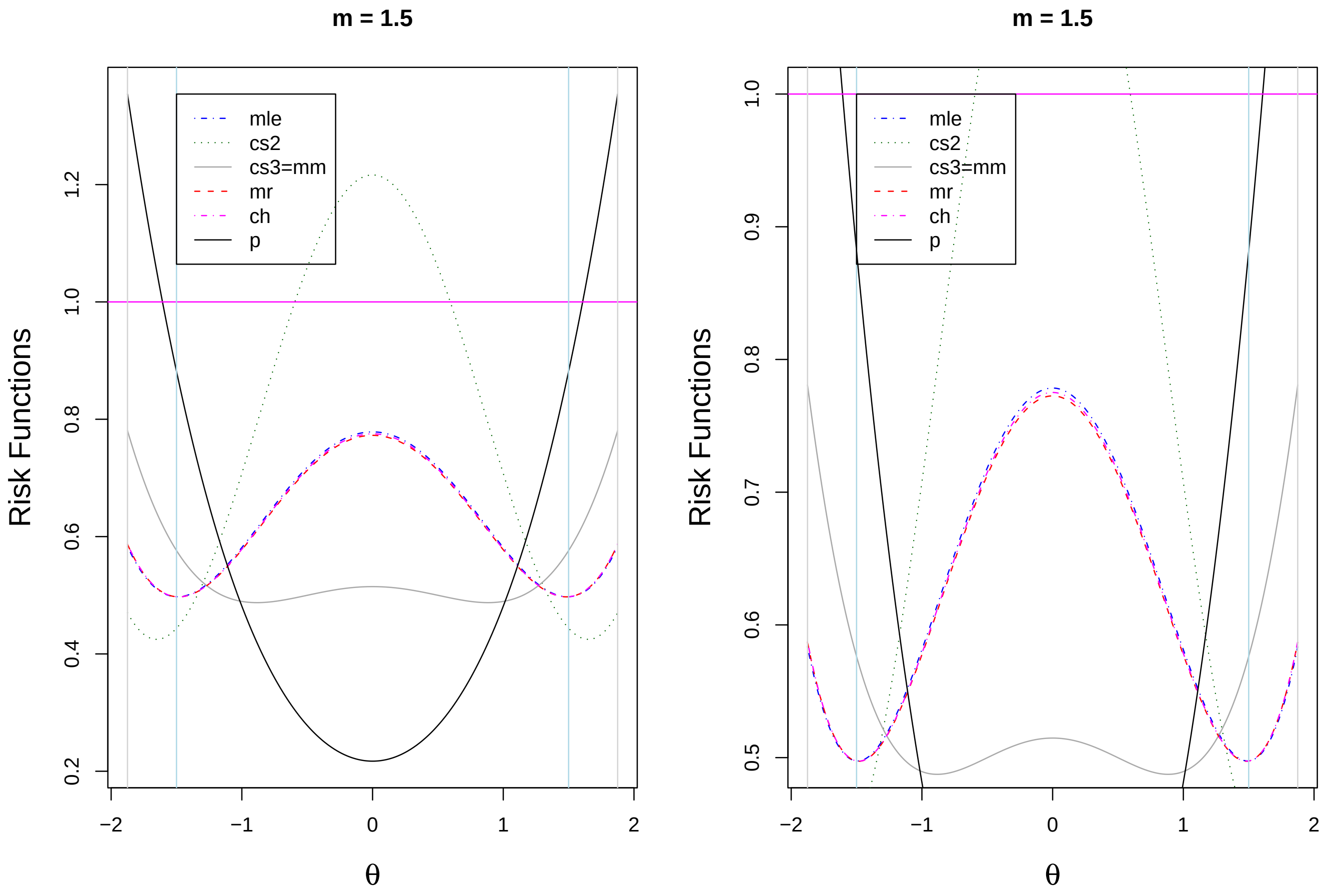

Figure 4: Comparisons of risk functions for $\delta^{\text{mle}}, \delta^{\text{cs.2}}, \delta^{\text{cs.3}}, \delta^{\text{mr}}, \delta^{\text{ch}}$ and δ^{P}, at $m = 1.5$.

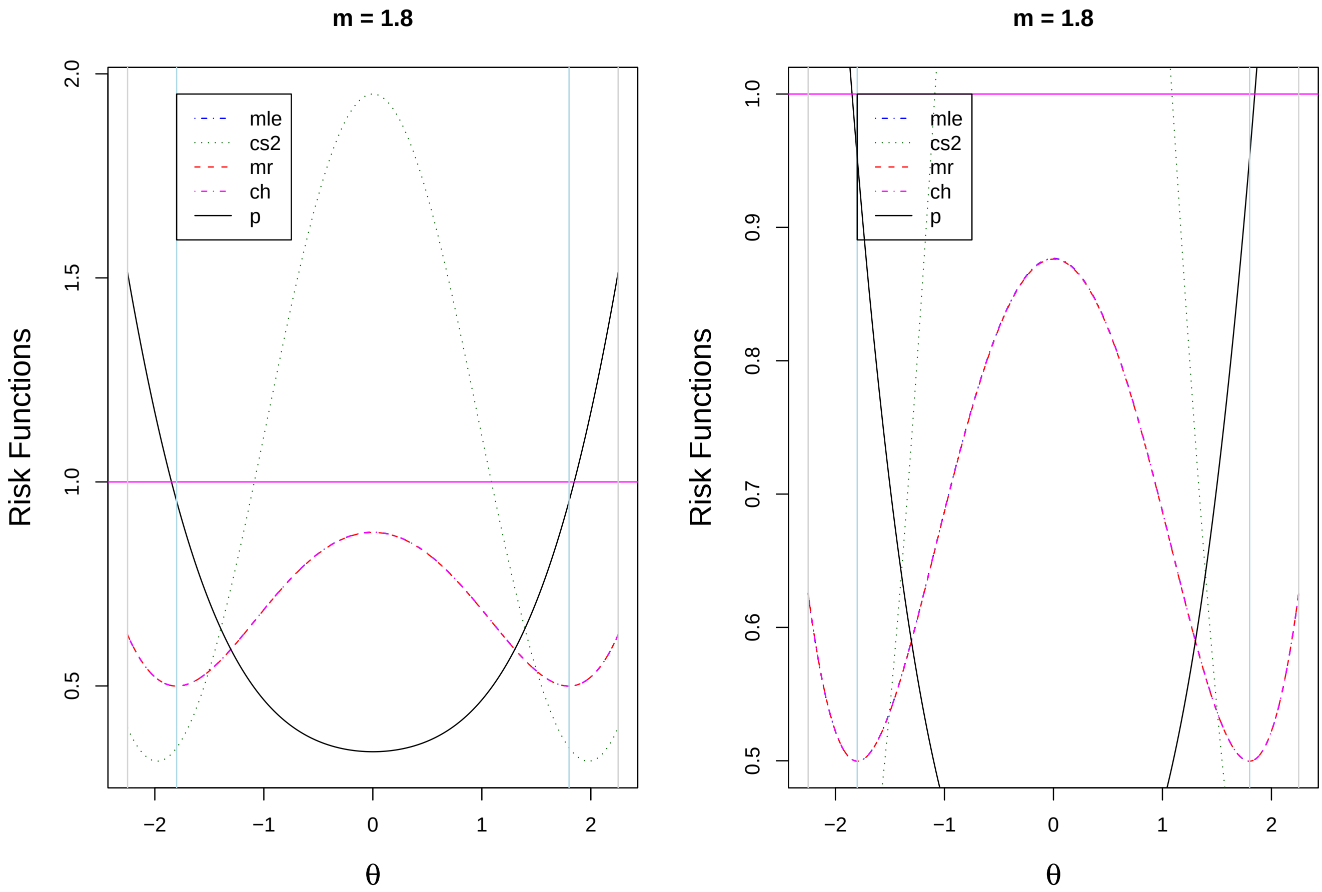

Figure 5: Comparisons of risk functions for $\delta^{\rm mle}, \delta^{\rm cs.2}, \delta^{\rm mr}, \delta^{\rm ch}$ and $\delta^{\rm P}$, at $m = 1.8$.

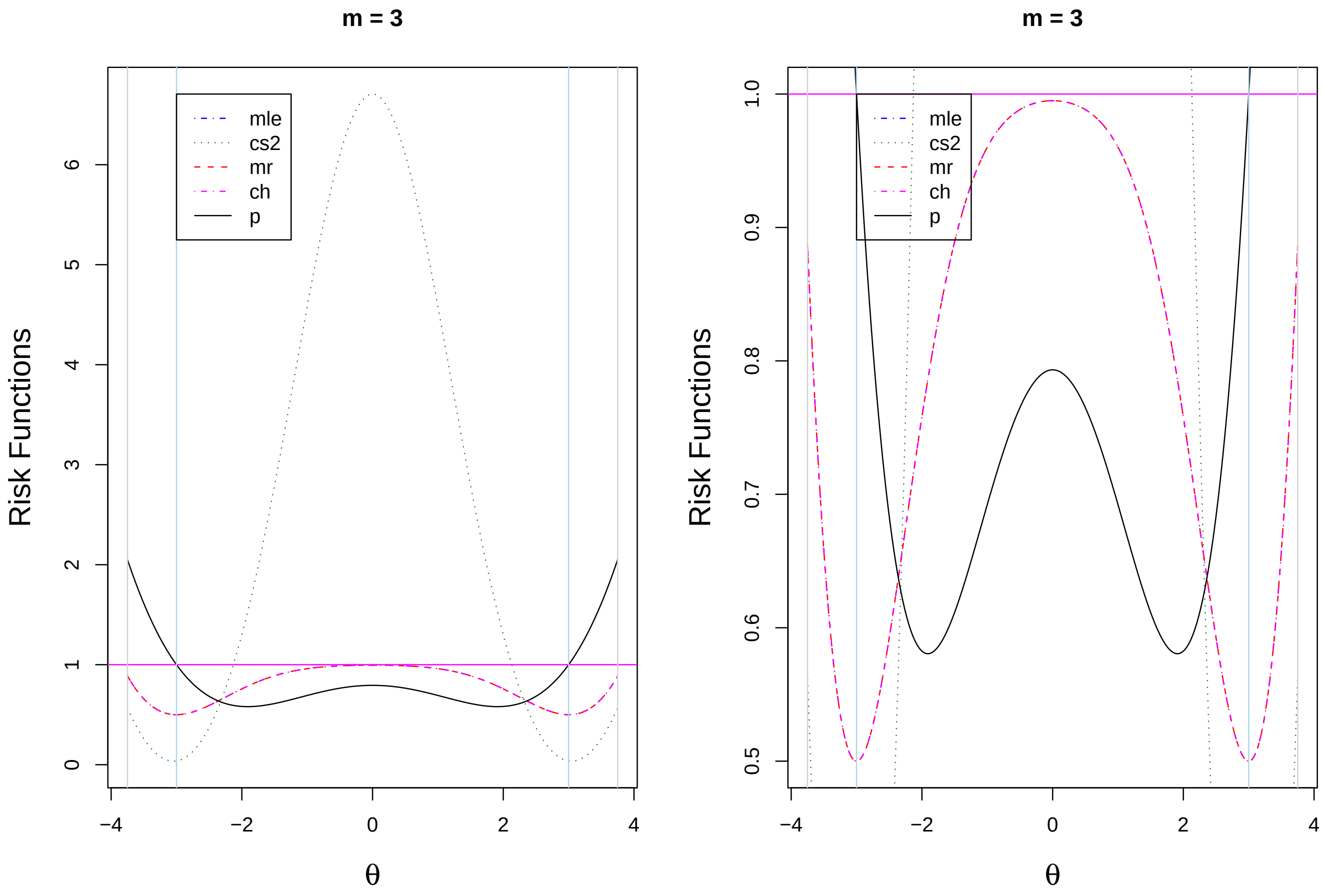

Figure 6: Comparisons of risk functions for $\delta^{\text{mle}}, \delta^{\text{cs.2}}, \delta^{\text{mr}}, \delta^{\text{ch}}$ and δ^{P}, at $m = 3$.

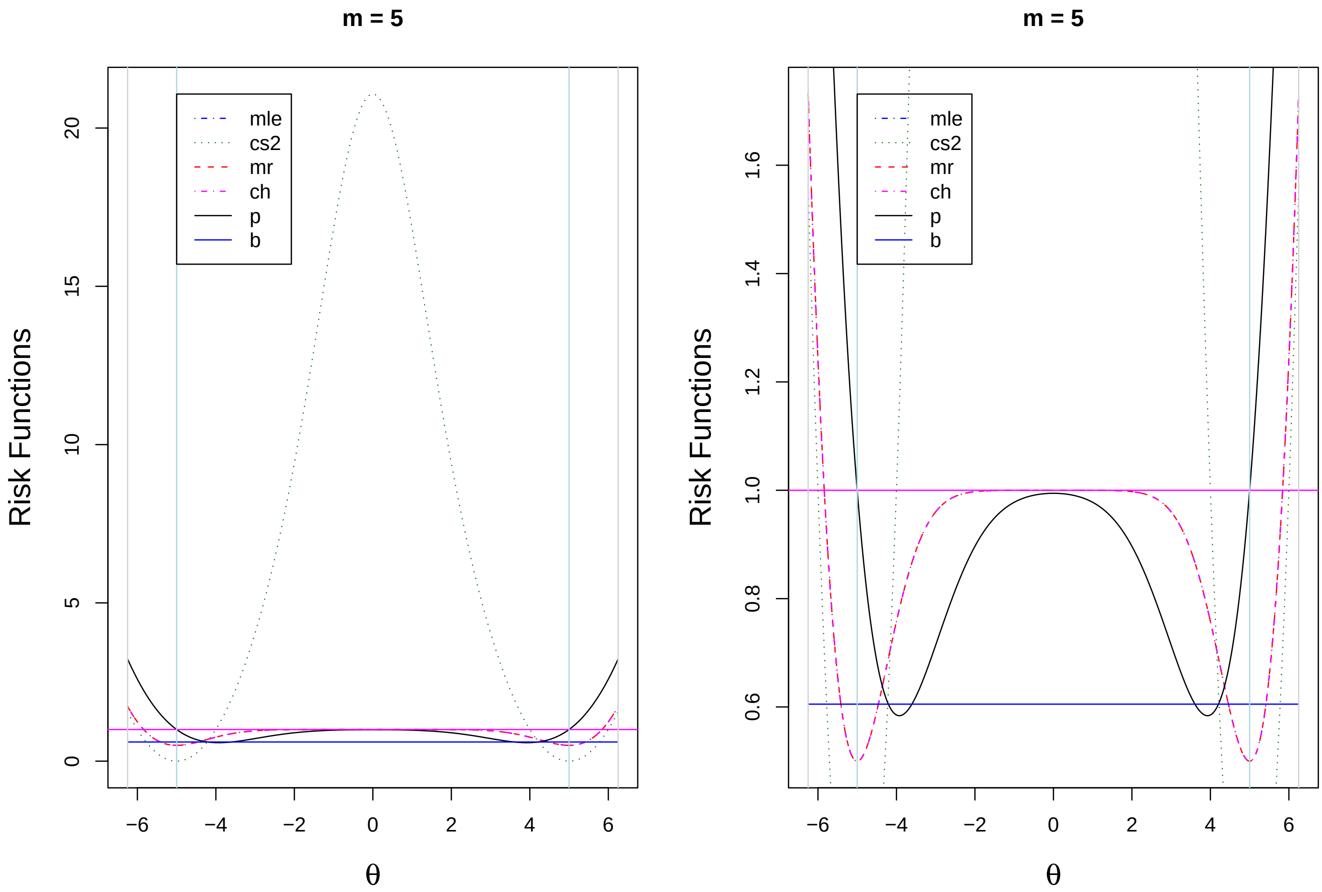

Figure 7: Comparisons of risk functions for $\delta^{\mathrm{mle}}, \delta^{\mathrm{cs.2}}, \delta^{\mathrm{mr}}, \delta^{\mathrm{ch}}, \delta^{\mathrm{P}}$ and δ^{B}, at $m = 5$.

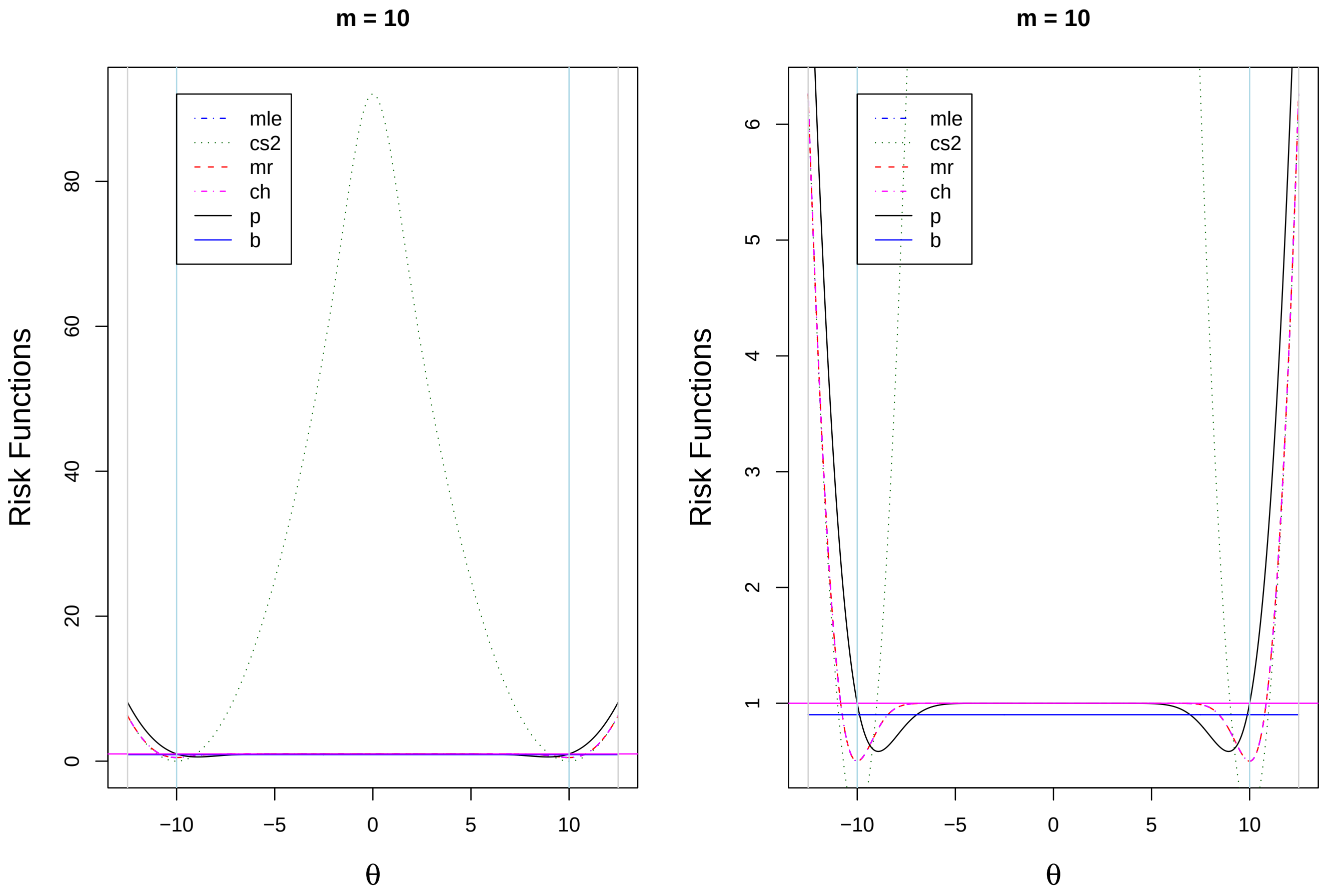

Figure 8: Comparisons of risk functions for $\delta^{\text{mle}}, \delta^{\text{cs.2}}, \delta^{\text{mr}}, \delta^{\text{ch}}, \delta^{\text{P}}$ and δ^{B}, at $m = 10$.